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Effect of random synaptic dilution in oscillator neural networks

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To discuss robustness against damage of synaptic connections in oscillator neural networks, we examine the effect of random dilution of the synapses in the case in which connections are symmetric. Deriving the mean-field equations in the replica symmetry approximation, we obtain the storage capacity and the critical overlap as functions of the ratio of the disconnected synapses. In addition, we find that theoretical results for this system are supported by our numerical simulations. Finally, it is suggested that the oscillator neural network is more robust against dilution than is the Hopfield model. [S1063-651X(97)12206-1]

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In recent years, oscillatory units have often been used to model neural networks. This trend was caused partially by some recent experimental results, which suggest that synchronization of pulses may contribute to information processing in neuronal systems [1]. From the theoretical point of view, there have been several attempts to explore the potential ability of such temporal coding in artificial neural networks [2]. What is important here is that utilizing temporal features of neuronal activities for the coding mechanism might enable the network to process information more effectively. However, these temporal features, including synchronization and timing of firings, cannot be described naturally by the McCulloch-Pitts formulation, because it is assumed that information is encoded only by the averaged firing rates of the neurons. As a first step toward modeling the temporal aspects of neural networks, we believe that the oscillator neural network provides a suitable framework.

Some theoretical results concerning oscillator neural networks have been obtained by several authors [3–10]. In particular, when the Hebbian rule with random patterns is used in an equivalent model, Cook estimated the storage capacity as $\alpha_c = 0.037$ using the replica symmetry approximation [6]. However, in comparison with the Hopfield model [11], there are many important unsolved problems with regard to oscillator neural networks. Among such problems, one we cannot avoid is that of theoretically estimating robustness against damage of the synaptic structure. In the case of the Hopfield model, this has already been studied by several authors [12–15]. Oscillator neural networks can retrieve more detailed information than can the Hopfield model, because the memorized patterns are described by continuous rather than binary variables. For this reason, one might guess that the retrieval ability of the oscillator neural network decreases faster with the ratio of the disconnected synapses than that of the Hopfield model. To clarify the above point, in this paper we wish to address the problem of how the oscillator neural network is affected by random synaptic dilution.

Let us start with a survey of the theoretical basis of oscillator neural networks. We first consider the situation in which N periodic firing neuronal systems are coupled with each other. In general, such a system can be described by evolution equations involving a set of state variables, for example, a membrane potential and several ionic leak currents. Since each system (a neuronal group or a single neuron) fires periodically in the absence of interactions, it can be regarded as a limit cycle oscillator. On the other hand, it is well known that, in the case of weak coupling, such a coupled system can be reduced to a system of simple coupled phase oscillators [16]. In this model, the state of the ith system can be characterized by a single variable ϕ_i. This quantity ϕ_i is usually referred to as the phase, which represents the timing of neuronal firings. Using such a reduction technique, we obtain the general form of the phase equations,

$$\frac{d\phi_i}{dt} = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\phi_j - \phi_i), \tag{1}$$

where ω_i is the frequency of the ith neuron, and Γ_{ij} represents the effect of the interaction between the ith and jth neurons. We should remark that the system is invariant under uniform phase translations $\phi_i \rightarrow \phi_i + \phi_0$, where ϕ_0 is an arbitrary constant. It has also been theoretically shown that $\Gamma_{ij}(\phi)$ is a 2π-periodic function of ϕ. To be specific, we must assume a certain form for each Γ_{ij} and ω_i. In this paper, we assume that $\omega_i = \Omega$ and $\Gamma_{ij} = J_{ij}\sin(\phi_j - \phi_i + \beta_{ij})$. Eliminating Ω by applying the transformation $\phi_i \rightarrow \phi_i + \Omega t$, we then obtain

$$\frac{d\phi_i}{dt} = \sum_{j=1}^{N} J_{ij}\sin(\phi_j - \phi_i + \beta_{ij}), \tag{2}$$

where J_{ij} and β_{ij} should be chosen so that the system is able to retrieve the stored patterns.

To see the similarity between this model and the traditional neural networks, it may be convenient to introduce the complex representation $W_i = \exp(i\phi_i)$ [7,8]. Using the expression W_i in Eq. (2), the equilibrium stable states of the system satisfy the conditions

$$W_i = \frac{h_i}{|h_i|}, \quad h_i = \sum_{j=1}^{N} C_{ij} W_j, \tag{3}$$

where $C_{ij} = J_{ij}\exp(i\beta_{ij})$. This can be regarded as the extended form of the traditional neural networks.

Let us denote a set of P patterns to be memorized as complex variables $\xi_i^\mu = \exp(i\theta_i^\mu)$ $(\mu = 1,2,\ldots,P)$, where θ_i^μ represents the phase value of the ith neuron in the μth pattern. For simplicity, we assume that the θ_i^μ are chosen at random from a uniform distribution between 0 and 2π. It must be noted that any pattern generated by the uniform phase translation $\xi_i^\mu \exp(i\phi_0)$ represents a pattern equivalent to ξ_i^μ. This is because information is encoded not by the absolute time but by the relative time of neuronal firings. To examine the effect of the random dilution, we suppose that the synaptic efficacies have the form

$$C_{ij} = J_{ij} \exp(i\beta_{ij}) = \frac{c_{ij}}{Nc} \sum_{\mu=0}^{P} \xi_i^\mu \tilde{\xi}_j^\mu, \tag{4}$$

where $\tilde{\xi}_i^\mu$ is the complex conjugate of ξ_i^μ. The dilution coefficients c_{ij} are independent random variables taking the values 1 and 0 with probabilities c and $1-c$, respectively. To obtain analytical results, we restrict ourselves to the symmetric diluted synapses, that is, we consider the case $c_{ij} = c_{ji}$. We briefly touch upon the effect of asymmetric dilution later. Note that Nc is equal to the average number of nonzero connections per neuron. Furthermore, following Sompolinsky's idea, in the limit $N \to \infty$ the synaptic matrix C_{ij} in Eq. (4) can be written as a fully connected model with synaptic noise,

$$C_{ij} = \frac{1}{N} \sum_{\mu=0}^{P} \xi_i^\mu \tilde{\xi}_j^\mu + \eta_{ij}. \tag{5}$$

Here, the synaptic noise η_{ij} is a complex random variable obeying Gaussian distribution with $[\eta_{ij}] = 0$ and $[|\eta_{ij}|^2] = \eta^2$. The square brackets [ ] denote averaging over the distribution of the noise. It is easily shown that the relationship between the variance of this noise and the dilution parameter c is given by

$$\eta^2 = \frac{(1-c)\alpha}{c}. \tag{6}$$

To apply statistical mechanics for the analysis of the equilibrium states, we introduce an additional noise term $\gamma_i(t)$ in the model represented by Eq. (2). Since the synaptic matrix C_{ij} in Eq. (4) is Hermitian, i.e., $C_{ij} = \tilde{C}_{ji}$, the model equations can be rewritten in the gradient forms

$$\frac{d\phi_i}{dt} = -\frac{\partial H}{\partial \phi_i} + \gamma_i(t), \quad H = -\tfrac{1}{2} \sum_{i \neq j} C_{ij} \tilde{W}_i W_j, \tag{7}$$

where the last term $\gamma_i(t)$ is Gaussian white noise characterized by $\langle \gamma_i(t) \rangle = 0$ and $\langle \gamma_i(t) \gamma_j(t') \rangle = 2T\delta_{ij}\delta(t-t')$. The temperature T $(=\beta^{-1})$ gives a measure of the level of the stochastic noise in the dynamics. The introduction of this noise enables us to perform the standard mean-field analysis in terms of statistical mechanics. Therefore, the asymptotic behavior of the network at finite T is governed by the free energy, and the equilibrium probability density is given by the Gibbs distribution $e^{-H/T}$. Using the synaptic matrix (5), the Hamiltonian H takes the form

$$H = -\frac{1}{2N} \sum_{\mu}^{P} \left[\left(\sum_i \cos(\phi_i - \theta_i^\mu) \right)^2 + \left(\sum_i \sin(\phi_i - \theta_i^\mu) \right)^2 \right] + \frac{P}{2} - \sum_{i<j} \eta_{ij}^{\mathrm{Re}} \cos(\phi_j - \phi_i) + \sum_{i<j} \eta_{ij}^{\mathrm{Im}} \sin(\phi_j - \phi_i), \tag{8}$$

where η_{ij}^{Re} and η_{ij}^{Im} are the real and the imaginary parts of η_{ij}, respectively.

We consider the retrieval states in which $m_1 = m \sim O(1)$ and $m_\mu \sim O(1/\sqrt{N})$ for $\mu > 1$. The overlap m_μ is defined as

$$m_p \equiv \frac{1}{N} \left| \sum_{j=1}^{N} \tilde{\xi}_j^\mu W_j \right| = \frac{1}{N} \left| \sum_{j=1}^{N} \exp(i\phi_j(t) - i\theta_j^p) \right|. \tag{9}$$

Then let us define the parameter α by $\alpha = P/N$. To proceed, we must first perform the quenched averaging of the free energy over the randomness. Using the replica method, the averaged free energy per neuron is computed from

$$f \equiv \lim_{N\to\infty} -\frac{1}{N\beta} \langle\langle \ln Z \rangle\rangle = \lim_{N\to\infty} \lim_{n\to 0} -\frac{1}{N\beta n} \ln \langle\langle Z^n \rangle\rangle, \tag{10}$$

where $\langle\langle \ \rangle\rangle$ indicates a quenched average over the patterns ξ_i^μ as well as over the synaptic noise η_{ij}. The partition function Z is defined by $Z = \mathrm{Tr}_{\{\phi_i\}} e^{-\beta H(\{\phi_i\})}$.

In the replica symmetry approximation, we find that the averaged free energy per neuron is given by

$$f = \frac{1}{2} m^2 + \frac{\alpha}{2} + \frac{\beta\alpha r}{8}(1-q) + \frac{\alpha}{\beta} \left\{ \ln\left[1 - \frac{1}{2}\beta(1-q)\right] - \frac{\beta\eta^2}{8}(1-q)^2 - \frac{\frac{\beta}{2} q}{1 - \frac{\beta}{2}(1-q)} \right\}$$
$$- \frac{1}{\beta} \left\langle \left\langle \int \int \frac{dz_1 dz_2}{2\pi} \exp\left(-\frac{z_1^2 + z_2^2}{2} \right) \ln \int_0^{2\pi} d\phi \, \exp\beta[A \cos\phi + B \sin\phi] \right\rangle \right\rangle_\theta, \tag{11}$$

with

$$A=\frac{\sqrt{\alpha r+2\eta^2 q}}{2}z_1+m\cos\theta,$$

$$B=\frac{\sqrt{\alpha r+2\eta^2 q}}{2}z_2+m\sin\theta. \tag{12}$$

The double angular brackets $\langle\langle\ \rangle\rangle_\theta$ denote an average over θ with the same distribution as θ_i^μ. The network is then characterized by order parameters m, r and q. The parameter m represents the overlap with the retrieved pattern ξ_i^1. The parameters q and r correspond to the Edwards-Anderson order parameter and the mean-square random overlap for the unretrieved patterns, respectively. The saddle-point equations for these order parameters are

$$m=\left\langle\left\langle\int\int\frac{dz_1dz_2}{2\pi}\exp\left(-\frac{z_1^2+z_2^2}{2}\right)\frac{I_1(\beta\sqrt{A^2+B^2})}{I_0(\beta\sqrt{A^2+B^2})}\frac{A\cos\theta+B\sin\theta}{\sqrt{A^2+B^2}}\right\rangle\right\rangle_\theta,$$

$$\beta(1-q)=\frac{2}{\sqrt{\alpha r+2\eta q}}\left\langle\left\langle\int\int\frac{dz_1dz_2}{2\pi}\exp\left(-\frac{z_1^2+z_2^2}{2}\right)\frac{I_1(\beta\sqrt{A^2+B^2})}{I_0(\beta\sqrt{A^2+B^2})}\frac{Az_1+Bz_2}{\sqrt{A^2+B^2}}\right\rangle\right\rangle_\theta,$$

$$r=\frac{2q}{\left(1-\frac{\beta}{2}+\frac{\beta}{2}q\right)^2}, \tag{13}$$

where $I_k(z)$ is the kth-order modified Bessel function, defined by

$$I_k(z)=\frac{1}{2\pi}\int_0^{2\pi}d\phi\ e^{z\cos\phi}\cos k\phi. \tag{14}$$

We are now ready to discuss the storage capacity α_c in the case of the random diluted synapses. In the limit of zero noise, $\beta^{-1}\rightarrow 0$, q tends to 1, and Eqs. (13) reduce to the equations

$$m=f_1\left(\frac{m}{\sqrt{\alpha r+2\eta^2}}\right),$$

$$r=2\left\{1-\frac{1}{\sqrt{\alpha r+2\eta^2}}f_2\left(\frac{m}{\sqrt{\alpha r+2\eta^2}}\right)\right\}^{-2}, \tag{15}$$

where f_1 and f_2 are defined by

$$f_1(y)=\int_0^{2\pi}d\varphi\int_0^\infty dR\frac{R(R\sin\varphi+2y)e^{-R^2/2}}{2\pi(R^2+4Ry\sin\varphi+4y^2)^{1/2}},$$

$$f_2(y)=\int_0^{2\pi}d\varphi\int_0^\infty dR\frac{R(R^2+2Ry\sin\varphi)e^{-R^2/2}}{2\pi(R^2+4Ry\sin\varphi+4y^2)^{1/2}}. \tag{16}$$

Note that the relationship between the dilution parameter c and the mean-square deviation of the synaptic noise η is given by Eq. (6). These equations always have a trivial solution $m=0$, which corresponds to a spin glass state ($q\neq 0$). For $\alpha<\alpha_c$, there also exists a solution, for which $m\neq 0$, corresponding to a retrieval state. This retrieval solution disappears discontinuously at α_c, where the overlap m jumps from the finite value m_c to zero, except for the case $c=0$.

In Fig. 1, the storage capacity obtained from numerical solutions of Eqs. (15) is plotted as a function of the ratio of the disconnected synapses. In the case of the fully connected network, that is, $c=1(\eta=0)$, we obtained that $\alpha_c=0.038$ and $m_c=0.90$. This result is essentially identical to those of the Q-state clock model in the limit $Q\rightarrow\infty$ estimated by Cook. In general, it is expected that α_c falls monotonically from 0.038 to zero as η increases. In fact, analytical results show that the retrieval solution exists only in the case that $\eta<\eta_c$, $\eta_c=\sqrt{\pi}/2\approx 0.886$, and α_c is a monotonically decreasing function of η. On the other hand, in case of the Hopfield model, Sompolinsky has estimated that $\eta_c=\sqrt{2/\pi}\approx 0.797$. Using Eq. (6), we finally obtained a theoretical curve, as shown in Fig. 1. We also carried out numerical simulations in which each value of α_c was averaged

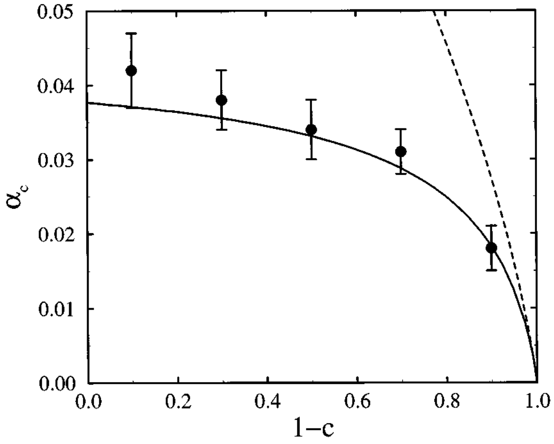

FIG. 1. Dependence of the storage capacity α_c on the ratio of disconnected synapses $1-c$. The solid curve represents theoretical results. The data points indicate simulation results with $N=1500$ for 20 trials. For reference, the theoretical results of the Hopfield model are indicated by the dashed curve.

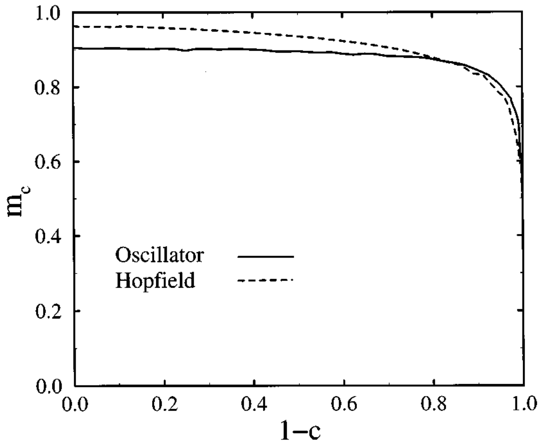

FIG. 2. Dependence of the critical overlap m_c on the ratio of disconnected synapses $1-c$. The solid and the dashed curves correspond to the oscillator and the Hopfield models, respectively.

over 20 trials with $N=1500$. As is clear from Fig. 1, the simulation results are in reasonable agreement with analytical results. Figure 2 shows the overlap of the retrieval state at α_c, where the solid and dashed curves correspond to the oscillator model and the Hopfield model, respectively. In either case, the critical overlap is affected little by the synaptic dilution as long as $1-c$ is smaller than 0.8. Particularly in the oscillator neural network, the critical overlap m_c remains almost constant in the range 0 to 0.8.

As mentioned above, η_c is slightly larger in the present model than in the Hopfield model. This implies that our system is more robust against synaptic dilution than is the Hopfield model. Let us attempt to clarify this point quantitatively. Although α_c in the oscillator network is generally smaller than in the Hopfield model [17], the oscillator network is able to retrieve phase patterns represented by continuous variables, not simple binary ones. Thus, taking account of the information content in the retrieved patterns, it makes no sense to compare the storage capacities α_c of the two models. However, it is meaningful to estimate and compare how the random dilution of synapses in each model reduces its performance from the level without dilution $(c=1)$. For this purpose, we define the normalized maximum storage capacity as $\alpha_c^*=\alpha_c/\alpha_c^0$, where α_c^0 is the maximum storage capacity at $c=1$. Thus, $\alpha_c^0=0.038$ in the oscillator model, and $\alpha_c^0=0.138$ in the Hopfield model. The dependence of the normalized storage capacities α_c^* on $1-c$ is shown in Fig. 3. It is obvious from this figure that the normalized capacity α_c^* of the oscillator network is always larger than that of the Hopfield network. Nevertheless, for $c<0.8$, the qualities of the retrieval patterns obtained in either model are largely independent of c, as seen in Fig. 2. Therefore, we can conclude that the oscillator network is totally more robust against dilution than the Hopfield model.

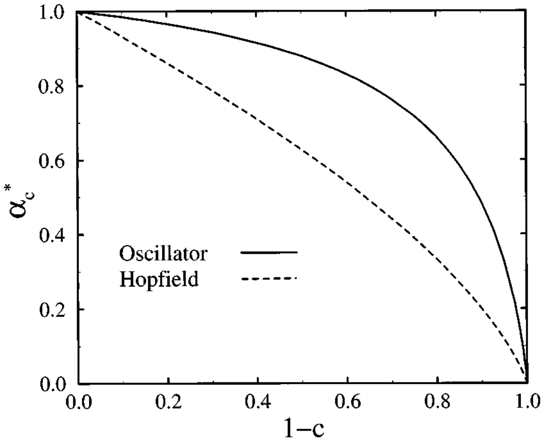

FIG. 3. Comparison of the normalized storage capacities, α_c^*, between the oscillator model and the Hopfield model.

In conclusion, using the replica symmetric solution, we have estimated the influence of random synaptic dilution on the storage capacity and the critical overlap in oscillator neural networks. As a result, the difference between the performance of the fully connected network and that for a diluted network is smaller in the case of the oscillator model than in the case of the Hopfield model. In this sense, we conclude that the performance of the oscillator network is superior to that of the Hopfield model. Since the oscillator network is also capable of retrieving more detailed information, this is an unexpected result. This suggests that the robustness against the damage of the synaptic connection, at least, is not deteriorated by utilizing the timing of the firings for the coding mechanism. This result encourages theoretical attempts to explore the potential ability of temporal coding.

Finally, we would like to make some comments. First, one may notice that the storage capacity found using simulations is slightly larger than that from theoretical results. This slight increase may be attributed to replica symmetry breaking. Second, we assumed that the symmetry of the connections is maintained in dilution process. However, this condition is not realistic from the biological point of view. It is important to study the effect of such asymmetric dilution [18,19]. In a preliminary study, we found that the results change little unless the ratio of the asymmetric connections is high. We will report details of this work at a later date.

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